Hypergeometric solutions for third order linear ODEs

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Abstract

In this paper we present a decision procedure for computing ${}_p\mathrm{F}_q$ hypergeometric solutions for third order linear ODEs, that is, solutions for the classes of hypergeometric equations constructed from the ${}_3\mathrm{F}_2$, ${}_2\mathrm{F}_2$, ${}_1\mathrm{F}_2$ and ${}_0\mathrm{F}_2$ standard equations using transformations of the form $x \to F(x)$, $y \to P(x)y$, where $F(x)$ is rational in x and $P(x)$ is arbitrary. A computer algebra implementation of this work is present in Maple 12.

Introduction

Given a third order linear ODE

$$y''' + c_2\, y'' + c_1\, y' + c_0\, y = 0 \tag{1}$$

where $y \equiv y(x)$ is the dependent variable and the $c_j \equiv c_j(x)$ are any functions of x such that the quantities[1]

$$I_1 = {c_2}' + \frac{{c_2}^2}{3} - c_1 \qquad I_0 = \frac{{c_2}''}{3} - \frac{2\,{c_2}^3}{27} + \frac{c_1 c_2}{3} - c_0 \tag{2}$$

are rational functions of x, the problem under consideration is that of systematically computing solutions for (1) even when no Liouvillian solutions exist[2]. Recalling, Liouvillian solutions can be computed systematically [1] and implementations of the related algorithm exist in various computer algebra systems. The linear ODEs involved in mathematical physics formulations, however, frequently admit only non-Liouvillian special function solutions, and for this case the existing algorithms cover a rather restricted portion of the problem.

The special functions associated with linear ODEs frequently happen to be particular cases of some generalized hypergeometric ${}_p\mathrm{F}_q$ functions [2]. One natural approach is thus to directly search for ${}_p\mathrm{F}_q$ solutions instead of special function solutions of one or another kind, and this is the approach discussed here. Related computer algebra routines were implemented in 2007 and are now at the root of the Maple (release 12) [3] ability for solving non-trivial 3rd order linear ODE problems.

The approach used consists of resolving an equivalence problem between a given equation of the form (1) and the four standard ${}_p\mathrm{F}_q$ differential equations associated to third order linear ODEs, that is, the ${}_3\mathrm{F}_2$, ${}_2\mathrm{F}_2$, ${}_1\mathrm{F}_2$ and ${}_0\mathrm{F}_2$ equations [4], respectively:

[1] I_1 and I_0 are invariant under transformations of the dependent variable of the form $y(x) \to P(x)\, y(x)$, P arbitrary.

[2] Expressions that can be expressed in terms of exponentials, integrals and algebraic functions, are called Liouvillian. The typical example is $\exp(\int R(x), dx)$ where $R(x)$ is rational or an algebraic function representing the roots of a polynomial.

$$
\begin{aligned}
&y''' - \frac{(\delta + \eta + 1 - (\alpha + \beta + \gamma + 3)\,x)}{x\,(x-1)}\,y'' - \frac{(\eta\,\delta - ((\beta+\gamma+1)\,\alpha + (\beta+1)\,(\gamma+1))\,x)}{x^2\,(x-1)}\,y' + \frac{\alpha\,\beta\,\gamma}{x^2\,(x-1)}\,y = 0\\
&y''' - \frac{(x - \gamma - \delta - 1)}{x}\,y'' - \frac{((\alpha+\beta+1)\,x - \gamma\,\delta)}{x^2}\,y' - \frac{\alpha\,\beta}{x^2}\,y = 0\\
&y''' + \frac{(\beta+\gamma+1)}{x}\,y'' - \frac{(x - \beta\,\gamma)}{x^2}\,y' - \frac{\alpha}{x^2}\,y = 0\\
&y''' + \frac{(\alpha+\beta+1)}{x}\,y'' + \frac{\alpha\,\beta}{x^2}\,y' - \frac{1}{x^2}\,y = 0
\end{aligned}
\tag{3}
$$

where $\{\alpha, \beta, \gamma, \delta, \eta\}$ represent arbitrary expressions constant with respect to x. The equivalence classes are constructed by applying to these equations the general transformation[3]

$$
x \to F(x),\ y \to P(x)\,y \tag{4}
$$

where $P(x)$ is arbitrary, with the only restriction that $F(x)$ is rational in x, resulting in rather general ODE families. When the equation being solved belongs to this class, apart from providing the values of $F(x)$ and $P(x)$ that resolve the problem, the algorithm systematically returns the values of the (five, four, three or two) ${}_p\mathrm{F}_q$ parameters entering each of the three independent solutions.

It is important to note that the idea of seeking hypergeometric function solutions for linear ODEs or using an equivalence approach for that purpose is not new, although in most cases the approaches presented only handle second order linear equations [6, 7, 8, 9]. An exception to that situation are the algorithms [10, 11] for computing ${}_p\mathrm{F}_q$ solutions for third and higher order linear ODEs, and a similar one implemented in Mathematica [12]. It is our understanding, however, that the transformations defining the classes of equivalence that those algorithms can handle are restricted to $x \to a x^b$, $y \to P(x)y$, with a and b constants, not having the generality of (4) with rational $F(x)$ presented here.

Apart from concretely expanding the ability to solve third order linear ODEs, the decision procedure being presented generalizes previous work in that:

1. The ideas presented in [9], useful for decomposing two sets of invariants into each other, were extended for third order equations and elaborated further.

2. The classification ideas presented in [9] for second order linear equations were extended for third order.

3. When the ${}_p\mathrm{F}_q$ parameters are such that less than three independent ${}_p\mathrm{F}_q$ solutions exist, instead of introducing integrals [11], MeijerG functions are used to express the missing independent solutions.

The combination of items 1 and 2 resulted in the new ability to solve the ${}_p\mathrm{F}_q$ ODE classes generated by transformations as general as (4) with $F(x)$ rational. Item 3 is not new[4], though we are not aware of literature presenting the related problem and solution. Altogether, these ideas and its related algorithm permit the systematic computation of three independent solutions for a large set of third order linear equations that we didn't know how to solve before.

1 Computing hypergeometric solutions

To compute ${}_p\mathrm{F}_q$ solutions to (1) the idea is to formulate an equivalence approach to the underlying hypergeometric differential equations, that is, to determine whether a given linear ODE can be obtained from one

[3]The problem of equivalence under transformations $\{x \to F(x),\ \ y \to P(x)\,y + Q(x)\}$ for linear ODEs can always be mapped into one with $Q(x) = 0$, see [5].

[4]Mathematica 6 also uses MeijerG functions as described in item 3.

of the ${}_pF_q$ ODEs (3) by means of a transformation of a certain type. If so, the solution to the given ODE is obtained by applying the same transformation to the solution of the corresponding ${}_pF_q$ equation.

The approach also requires determining the values of the hypergeometric parameters $\{\alpha, \beta, \gamma, \delta, \eta\}$ for which the equivalence exists, and it is clear that the bottleneck in this approach is the generality of the class of transformations to be considered. For instance, one can verify that for linear transformations of the form (4) with arbitrary $F(x)$, in the case of second order linear ODEs, the problem is too general in that the determination of $F(x)$ requires solving the given ODE itself [13], making the approach of no practical use. This has to do with the fact that in the second order case, any linear ODE can be obtained from any other one through a transformation of the form (4). The situation for third order equations is different: the transformation (4) is not enough to map any equation into any other one[5] [14], so that its determination when the equivalence exists is in principle possible. By restricting the form of $F(x)$ entering (4) to be rational in x the problem becomes tractable by using a two step strategy:

1. Compute a rational transformation $R(x)$ mapping the normal form of the given equation[6] into one having *invariants with minimal degrees* (defined in sec. 3).

2. Resolve an equivalence problem between this equation with minimal degrees and the standard ${}_pF_q$ equations (3) under transformations of the form discussed in [9], that is

$$x \to \frac{(a\,x^k + b)}{(c\,x^k + d)}, \qquad y \to P(x)\,y \tag{5}$$

   with $P(x)$ arbitrary and $\{a, b, c, d, k\}$ constants with respect to x. In doing so, determine also the parameters $\{\alpha, \beta, \gamma, \delta, \eta\}$ of the ${}_pF_q$ or MeijerG functions entering the three independent solutions.

The key observation in this "two steps" approach is that a transformation of the form (4) with rational $F(x)$ mapping into the ${}_pF_q$ equations (3), when it exists, it can always be expressed as the composition of two transformations, each one related to each of the two steps above (see sec. 3), because (3) have invariants with minimal degrees. The advantage of splitting the problem in this way is that the determination of $R(x)$ in step one, and of the (up to five) ${}_pF_q$ parameters in step two, as well as of the values of $\{a, b, c, d, k\}$ entering (5), is systematic (see sec. 2 and sec. 3), even when the problem is nonlinear in many variables.

2 Equivalence under $x \to (a\,x^k + b)/(c\,x^k + d)$, $y \to P(x)\,y$

This type of equivalence is discussed in [9] and generalized here for third order ODEs. Recalling the main points, these transformations, which do not form a group in the strict sense, can be obtained by sequentially composing three different transformations, each of which does constitute a group. The sequence starts with linear fractional - also called Möbius - transformations

$$x \to \frac{a\,x + b}{c\,x + d}, \tag{6}$$

is followed by power transformations

$$x \to x^k, \tag{7}$$

and ends with linear homogeneous transformations of the dependent variable

$$y \to P\,y. \tag{8}$$

[5]Therefore there exist enough absolute invariants to formulate the equivalence problem under (4) - see sec. 3.

[6]The coefficients of y' and y in the normalized equation are the invariants I_1 and I_0 defined in (2), assumed to be rational.

2.1 Equivalence under transformations of the dependent variable $y \to P(x)y$

Transformations of the form (8) can easily be factored out of the problem: if two equations of the form (1) can be obtained from each other by means of (8), the transformation relating them is computable directly from these coefficients. For that purpose first rewrite both equations in normal form using

$$y \to y\, e^{-\int c_2(x)/3\, dx} \tag{9}$$

and the transformation relating the two hypothetical ODEs - say with coefficients c_j and $\tilde{c}_k$, when it exists, is given by $y \to y\, e^{\int (c_2(x) - \tilde{c}_2(x))/3\, dx}$.

2.2 Equivalence under Möbius transformations, singularities and classification

Möbius transformations preserve the structure of the singularities of (1). For example, all of the ${}_0\mathrm{F}_2$, ${}_1\mathrm{F}_2$ and ${}_2\mathrm{F}_2$ hypergeometric equations in (3) have one regular singularity at the origin and one irregular singularity at infinity, and after transforming them using the Möbius transformations (6), they continue having one regular singularity and one irregular singularity, now respectively located at[7] $-b/a$ and $-d/c$.

In the case of the ${}_3\mathrm{F}_2$ differential equation (the first listed in (3)), under (6) the three regular singularities move from $\{0, 1, \infty\}$ to $\{-b/a,\ -d/c,\ (d-b)/(a-c)\}$. So from the singularities of an ODE, not only one can tell with respect to which of the four differential equations (3) could the equivalence under (6) be resolved, but also one can extract the values of the parameters $\{a, b, c, d\}$ entering the transformation (6).

More generally, through Möbius transformations one can formulate a classification of singularities of the linear ODEs "equivalent" to the third order ${}_p\mathrm{F}_q$ equations (3) as done in [9] for second order ${}_p\mathrm{F}_q$ equations. So, for each ${}_p\mathrm{F}_q$ family obtained from (3) using (6), a classification table can be constructed based only on:

- the degrees of the numerators and denominators of the invariants (2);
- the presence of roots with multiplicity in the denominators;
- the possible cancellation of factors between the numerator and denominator of each invariant.

With this classification in hands, from the knowledge of the degrees with respect to x of the numerator and denominator of the invariants (2) of a given third order linear ODE, one can determine systematically whether or not the equation could be obtained from the ${}_3\mathrm{F}_2$, ${}_2\mathrm{F}_2$, ${}_1\mathrm{F}_2$ or ${}_0\mathrm{F}_2$ equations (3) using (6).

2.3 Transformations $x \to F(x)$ and equivalence under $x \to x^k$

Changing $x \to F(x)$ in (1), the new invariants $\tilde{I}_j$ can be expressed in terms of the invariants (2) of (1) by

$$\begin{aligned} \tilde{I}_1(x) &= {F'}^2 I_1(F) - 2\, S(F) \\ \tilde{I}_0(x) &= F' F'' I_1(F) + {F'}^3 I_0(F) - S(F)' \end{aligned} \tag{10}$$

where $S(F)$ is the Schwarzian [15]

$$S(F) = \frac{F'''}{F'} - \frac{3}{2}\left(\frac{F''}{F'}\right)^2. \tag{11}$$

The form of $S(F)$ is particularly simple when $F(x)$ is a Möbius transformation, in which case $S(F) = 0$.

Regarding power transformations $F(x) = x^k$, unlike Möbius transformations, they *do not preserve* the structure of singularities; the Schwarzian (11) is:

$$S(x^k) = \frac{1 - k^2}{2\, x^2}. \tag{12}$$

From (10) and (12), for instance the transformation rule for $I_1(x)$ becomes

[7] When either a or c are equal to zero, the corresponding singularity is located at ∞

$$x^2\tilde{I}_1(x)+1 = k^2\left((x^k)^2 I_1(x^k)+1\right). \tag{13}$$

Generalizing to third order the presentation of shifted invariants in [9], we define here

$$\begin{aligned} J_1(x) &= x^2 I_1(x)+1,\\ J_2(x) &= x^3 I_0(x) + x^2 I_1(x). \end{aligned} \tag{14}$$

From (10) rewritten in terms of these $J_n(x)$, their transformation rule under $x \to x^k$ is given by

$$\begin{aligned} \tilde{J}_1(x) &= k^2 J_1(x^k),\\ \tilde{J}_2(x) &= k^3 J_2(x^k). \end{aligned} \tag{15}$$

The equivalence of two linear ODEs A and B under $x \to x^k$ can then be formulated as follows: Given the shifted invariants $\tilde{J}_{n,A}(x)$ and $\tilde{J}_{n,B}(x)$, computed using their definition (14) in terms of $\tilde{I}_n(x)$ defined in (2), compute k_A and k_B entering (15) such that the degrees of $J_{n,A}(x)$ and $J_{n,B}(x)$ are minimal. From the knowledge of $x \to x^{k_A}$ and $x \to x^{k_B}$, respectively leading to $J_{n,A}$ and $J_{n,B}$ with minimized degrees, equations A and B are related through power transformations only when $J_{n,A} = J_{n,B}$ and, if so, the mapping relating A and B is $x \to x^{k_A - k_B}$. Finally, the computation of k simultaneously minimizing the degrees of the two $J_n(x)$ in (15) is performed as explained in section 3 of [9].

3 Mapping into equations having invariants with minimal degrees

The decision procedure presented in the previous section serves for systematically solving well defined families of ${}_p\mathrm{F}_q$ 3rd order equations for which no solving algorithm was available before to the best of our knowledge. However, the restriction in the form of $F(x)$ entering (4) to the composition of Möbius with power transformations is unsatisfactory: for linear equations of order higher than two, (4) does not map any linear equation into any other one of the same order and so the problem is already restricted[8].

As shown in what follows, one possible extension of the algorithm is thus to consider the general transformations (4) restricting $F(x)$ to be a rational function of x. For that purpose, instead of working with invariants I_j under $y \to P(x)\,y$ we introduce absolute invariants L_i under $\{x \to F(x),\ y \to P(x)\,y\}$:

$$L_1 = \frac{(6rr'' + 9I_1r^2 - 7r'^2)^3}{r^8}, \quad L_2 = \frac{(27{I_1}'r^3 - 18I_1r^2r' + 56r'^3 - 72r''r'r + 18r'''r^2)}{r^4}; \tag{16}$$

where $r = {I_1}' - 2I_0$ is a relative invariant of weight 3 [16]. Under (4), L_i transforms as $L_i(x) \to L_i(F(x))$ and (16) can be inverted using as intermediate variables the relative invariants $s = (L_2L_1)/{L_1}'$, and $t = L_1/s^3$:

$$I_1 = \frac{st^3 - 6t''t + 7t'^2}{9t^2}, \quad I_0 = \frac{(s'-9)t^4 + t'st^3 - 6t'''t^2 + 20t''t't - 14t'^3}{18t^3}. \tag{17}$$

Thus, any canonical form for the L_i that can be achieved using (4) automatically implies on a canonical form for the I_i and so for the ODE (1). The canonical form we propose here is one where the L_i have *minimal degrees*, that is, where the maximum of the degrees of the numerator and denominator in each of the $L_i(x)$ is the minimal one that can be obtained using a rational transformation $x \to F(x)$. This canonical form is not unique in that it is still possible to perform a Möbius transformation (6), that changes the L_i but not their degrees.

The equivalence of two linear ODEs A and B under (4) with rational $F(x)$ can then be formulated by rewriting both equations in this canonical form, where the invariants L_i of each equation have minimal degrees, followed by determining whether these canonical forms are related through a Möbius transformation.

[8]The equivalence problem for linear equations of order n involves a system of $n-1$ equations and invariants $I_j(x)$, that includes the equivalence function $F(x)$. When $n > 2$, eliminating $F(x)$ from the problem results in an interrelation between the I_j so that the equivalence is only possible when these relationships between the I_j hold [14].

In the framework of this paper, B is one of the hypergeometric equations (3), all of them already in canonical form in that the corresponding L_i already have minimal degrees. Hence, the equivalence of A, of the form (1), and any of B of the form (3) requires determining only a canonical form for A (the rational function $F(x)$ minimizing the degrees of the L_i of A), followed by resolving an equivalence under Möbius transformations between this canonical form and any of the equations (3), done as explained in sec. 2.2.

The key computation in this formulation of the equivalence problem under (4) is thus the computation of a rational $F(x)$ that minimizes the degrees of the L_i of A. The computation of $F(x)$ can clearly be formulated as a rational function decomposition problem subject to constraints: *"given two rational functions $L_i(x), i = 1..2$, find rational functions $\tilde{L}_i(x)$ and $F(x)$ satisfying $L_i = \tilde{L}_i \circ F$ and such that the rational degree of F is maximized"* (and therefore the degrees of the canonical invariants $\tilde{L}_i$ are minimized). In turn, this type of function decomposition associated to "minimizing the degrees" of the L_i can be interpreted as the reparametrization, in terms of polynomials of lower degree, of a rational curve that is improperly parameterized, as discussed in [17], where an algorithm to perform this reparametrization is presented.

One key feature of the algorithm presented in [17] is that it reduces the computation of $F(x)$ to a sequence of *univariate* GCD computations, avoiding the expensive computation of bivariate GCD. However, it is not clear for us whether the prescriptions in [17] (at page 71) for mapping the bivariate GCDs into univariate ones is complete. We also failed in obtaining a copy of the computer algebra packages *FRAC* [18] or *Cadecom* [19] that contain an implementation of the algorithm presented in [17]. Mainly for these reasons, and without the intention of being original, we describe here a slightly modified version of the algorithm presented in [17].

3.1 An algorithm for computing $x \to F(x)$ minimizing the degrees of the $L_i(x)$

Let $L_i(x) = \tilde{L}_i(F(x)) = N_i(x)/D_i(x), i = 1..n$, and $F(x) = p(x)/q(x)$, where the $\tilde{L}_i$ have minimal degrees, N_i is relatively prime to D_i and p is relatively prime to q. Construct polynomials

$$Q_i(x,t) = \text{numerator}(L_i(x) - L_i(t)) = N_i(x)D_i(t) - N_i(t)D_i(x), \tag{18}$$

and let $P(x,t)$ be the bivariate GCD of these $Q_i(x,t)$. Consequently

$$P(x,t) = \text{numerator}(F(x) - F(t)) = \sum_i P_i(x)\, t^i = p(x)\, q(t) - p(t)\, q(x), \tag{19}$$

The coefficient $P_i(x)$ of each power of t in $P(x,t)$ is a linear combination of $p(x)$ and $q(x)$, and because the quotient of any two relatively prime of these linear combinations is fractional linear in $F(x)$, so is the quotient of any two relatively prime $P_i(x)$. Finally, because $F(x)$ is defined up to a Möbius transformation we can take that quotient itself - say, $P_i(x)/P_j(x)$ - as the solution $F(x)$.

The slowest step of this algorithm is the computation of the bivariate GCD between the $Q_i(x,t)$ that determines the function $P(x,t)$ from which the $P_i(x)$ are computed. It is possible however to avoid computing that bivariate GCD, using a small number of univariate GCD computations instead.

For that purpose, notice first that what is relevant in the $P_i(x)$ is that they are linear combinations of $p(x)$ and $q(x)$. Now, we can also obtain linear combinations of $p(x)$ and $q(x)$ by directly substituting numerical values t_k for t into $P(x,t)$, and from there compute $F(x)$ as the quotient, e.g., of $P(x,t_0)/P(x,t_1)$. The key observation here is that these $P(x,t_k)$ can also be obtained by substituting $t = t_k$ directly into the $Q_i(x,t)$ followed by computing the univariate GCD of $Q_1(x,t_k)$ and $Q_2(x,t_k)$[9], avoiding in this way the computation of the expensive bivariate GCD leading to $P(x,t)$.

Repeating this process with another t-value gives a second, in general different, such linear combination of $p(x)$ and $q(x)$, with F being the resulting quotient of two of these linear combinations obtained using different values of t. The rest of the algorithm entails avoiding invalid t-values at the time of substituting $t = t_k$ and this is accomplished by considering different t_k until the following conditions are both satisfied:

[9]For example, suppose the x-solutions of $P(x,t) = 0$ are $x = X_j(t), j = 1..m$, i.e., $P(x,t) = -P_m(t)\prod_j x - X_j(t)$. Then each $x = X_j(t_0)$ is a solution of both $Q_1(x,t_0) = 0$ and $Q_2(x,t_0) = 0$. For most values of t_0 (all but a finite set in fact) these $X_j(t_0)$ will be the only such common solutions, and therefore the GCD of $Q_1(x,t_0)$ and $Q_2(x,t_0)$ is in fact $P(x,t_0)$.

1. The two $P(x,t_0)$, $P(x,t_1)$ whose quotient gives the solution $F(x)$ must be relatively prime.
2. The degree of F must divide the degrees of each $L_i, i = 1..n$.

4 Summary of the ${}_p\mathrm{F}_q$ approach for third order linear ODEs

The idea consists of assuming that the given linear ODE is one of ${}_p\mathrm{F}_q$ equations (3) transformed using (4) for some $F(x)$ rational in x and $P(x)$ arbitrary and for some values of the pFq parameters. Resolving the equivalence is about determining the $F(x)$, $P(x)$ and the values of the ${}_p\mathrm{F}_q$ parameters $\{\alpha, \beta, \gamma, \delta, \eta\}$ such that the equivalence exists. An itemized description of the decision procedure to resolve this equivalence, following the presentation the previous sections, is as follows.

1. Rewrite the given equation (1) we want to solve, in normal form

$$y''' = \tilde{I}_1(x)\, y' + \tilde{I}_0(x)\, y \tag{20}$$

   where the invariants $\tilde{I}_n(x)$ are constructed using the formulas (2).

2. Verify whether an equivalence of the form $\{x \to (a\, x^k + b)/(c\, x^k + d),\ y \to P(x)\, y\}$ exists:
   (a) Compute $\tilde{J}_n(x)$, the shifted invariants (14), and use transformations $x \to x^k$ to reduce to the integer minimal values the powers entering the numerator and denominator; i.e., compute k and $J_n(x)$ in (15).
   (b) Determine the singularities of the $J_n(x)$ and use the classification of singularities mentioned in section 2 to tell whether an equivalence under Möbius transformations to any of the ${}_3\mathrm{F}_2$, ${}_2\mathrm{F}_2$, ${}_1\mathrm{F}_2$ or ${}_0\mathrm{F}_2$ equations (3) exists.
   (c) When the equivalence exists, from the singularities of the two $J_n(x)$ compute the parameters $\{a, b, c, d\}$ entering the Möbius transformation (6) as well as the hypergeometric parameters $\{\alpha, \beta, \gamma, \delta, \eta\}$ entering the ${}_p\mathrm{F}_q$ equation (3).
   (d) Compose the three transformations to obtain one of the form

$$x \to \frac{\alpha x^k + \beta}{\gamma x^k + \delta}, \quad y \to P(x)\, y$$

   mapping the ${}_p\mathrm{F}_q$ equation involved into the ODE being solved.

3. When the equivalence of the previous step does not exist, perform step 1 in the itemization of section 1, that is, compute the absolute invariants L_i (16) and compute a rational transformation $R(x)$ minimizing the degrees of the invariants (16) of the given equation
   (a) When $R(x)$ is not of Möbius form, change $x \to R(x)$ rewriting the given equation in canonical form and re-enter step (2) with it, to resolve the remaining Möbius transformation and determining the values of the ${}_p\mathrm{F}_q$ parameters.

4. When either of the equivalences considered in steps (2) or (3) exist, compose all the transformations used and apply the composition to the known solution of the ${}_p\mathrm{F}_q$ equation to which the equivalence was resolved, obtaining the solution to the given ODE.

5 Special cases and MeijerG functions

Giving a look at the series expansion of any of the ${}_3\mathrm{F}_2$, ${}_2\mathrm{F}_2$, ${}_1\mathrm{F}_2$ or ${}_0\mathrm{F}_2$ functions one can see that there are some different situations that require special attention at the time of constructing the three independent solutions to (1). Consider for instance the standard ${}_0\mathrm{F}_2$ equation and its three independent solutions,

$$\begin{gathered} y''' + \frac{(\alpha+\beta+1)}{x}\, y'' + \frac{\alpha\,\beta}{x^2}\, y' - \frac{1}{x^2}\, y = 0 \\ y = {}_0\mathrm{F}_2(\ ;\, \alpha, \beta;\, x)\, C_1 + x^{1-\beta}{}_0\mathrm{F}_2(\ ;\, 2-\beta, 1+\alpha-\beta;\, x)\, C_2 + x^{1-\alpha}{}_0\mathrm{F}_2(\ ;\, 2-\alpha, 1-\alpha+\beta;\, x)\, C_3 \end{gathered} \tag{21}$$

where the C_i are arbitrary constants. Expanding in series the first ${}_0\mathrm{F}_2$ function entering this solution we get

$$1 + \frac{1}{\alpha\,\beta}x + \frac{1}{2\,\alpha\,\beta\,(\alpha+1)\,(1+\beta)}x^2 + \frac{1}{6\,\alpha\,\beta\,(\alpha+1)\,(1+\beta)\,(\alpha+2)\,(\beta+2)}x^3 + O\left(x^4\right) \tag{22}$$

This series does not exist when α or β are zero or negative integers, and the same happens when the ${}_p\mathrm{F}_q$ parameters entering any of the other two independent solutions is a non-positive integer. By inspection, however, one of the three ${}_p\mathrm{F}_q$ functions entering the solution in (21) always exists, because there are no α and β such that the three ${}_0\mathrm{F}_2$ functions simultaneously contain non-positive integer parameters.

Consider now the second independent solution, $x^{1-\beta}{}_0\mathrm{F}_2(\ ;\, 2-\beta, 1+\alpha-\beta;\, x)$: when $\beta = 1$ it becomes equal to the first one and so we have only two independent ${}_p\mathrm{F}_q$ solutions. In the same way, when $\alpha = 1$ the first and third solutions entering (21) are the same and when $\alpha = \beta$ the second and third solutions are the same. And when the two conditions hold, that is $\alpha = \beta = 1$, actually the three solutions are the same. Notwithstanding, in these cases too one of the three ${}_0\mathrm{F}_2$ solutions always exists.

The same two type of special cases exist for the ${}_1\mathrm{F}_2$, ${}_2\mathrm{F}_2$ and ${}_3\mathrm{F}_2$ function solutions and the problem at hand consists of having a way to represent the three independent solutions to (1) *without* introducing integrals or iterating reductions of order[10]. For this purpose, we use a set of 3 MeijerG functions for each of the four ${}_p\mathrm{F}_q$ families that can be used to replace the missing ${}_p\mathrm{F}_q$ solutions in these special cases. The key observation is that at these special values of the last two parameters of the ${}_p\mathrm{F}_q$ functions the MeijerG replacements exist, satisfy the same differential equation and are independent of the available ${}_p\mathrm{F}_q$ function solutions. A table with these 3 x 4 = 12 MeijerG function replacements is as follows:

Table 1: MeijerG alternative solutions to the ${}_p\mathrm{F}_q$ equations

${}_p\mathrm{F}_q$ family	MeijerG functions		
${}_0\mathrm{F}_2(\ ;\, \alpha, \beta;\, x)$	$G^{2,0}_{0,3}\left(x, \left\vert {}_{0,1-\alpha,1-\beta} \right.\right)$	$G^{3,0}_{0,3}\left(-x, \left\vert {}_{0,1-\alpha,1-\beta} \right.\right)$	$G^{2,0}_{0,3}\left(x, \left\vert {}_{1-\alpha,1-\beta,0} \right.\right)$
${}_1\mathrm{F}_2(\alpha;\, \beta, \gamma;\, x)$	$G^{2,1}_{1,3}\left(x, \left\vert {}^{1-\alpha}_{0,1-\beta,1-\gamma} \right.\right)$	$G^{3,1}_{1,3}\left(-x, \left\vert {}^{1-\alpha}_{0,1-\gamma,1-\beta} \right.\right)$	$G^{2,1}_{1,3}\left(x, \left\vert {}^{1-\alpha}_{1-\gamma,1-\beta,0} \right.\right)$
${}_2\mathrm{F}_2(\alpha, \beta;\, \delta, \gamma;\, x)$	$G^{2,2}_{2,3}\left(x, \left\vert {}^{1-\beta,1-\alpha}_{0,1-\gamma,1-\delta} \right.\right)$	$G^{3,2}_{2,3}\left(-x, \left\vert {}^{1-\beta,1-\alpha}_{0,1-\gamma,1-\delta} \right.\right)$	$G^{2,2}_{2,3}\left(x, \left\vert {}^{1-\beta,1-\alpha}_{1-\gamma,1-\delta,0} \right.\right)$
${}_3\mathrm{F}_2(\alpha, \beta, \gamma;\, \delta, \eta;\, x)$	$G^{2,3}_{3,3}\left(x, \left\vert {}^{1-\beta,1-\alpha,1-\gamma}_{0,1-\delta,1-\eta} \right.\right)$	$G^{3,3}_{3,3}\left(-x, \left\vert {}^{1-\beta,1-\alpha,1-\gamma}_{0,1-\delta,1-\eta} \right.\right)$	$G^{2,3}_{3,3}\left(x, \left\vert {}^{1-\beta,1-\alpha,1-\gamma}_{1-\delta,1-\eta,0} \right.\right)$

6 Examples

Equivalence under power composed with Möbius transformations for the ${}_0\mathrm{F}_2$ class

Consider the third order linear ODE

[10] Recall that given two independent solutions, it is always possible to write the third one in terms of integrals constructed with the two existing solutions, and in the case of a single solution it is still possible to reduce the order to a second order linear equation that may or not be solvable.

$$
\begin{aligned}
y''' = {} & \frac{\left(37+2\,\mu+6\,\nu-108\,x^2\right)}{12\,x\,(x+1)\,(x-1)}\,y'' \\
& + \frac{\left(2\,(\nu+6)\,(11/2-\mu)+(36\,\nu+294+12\,\mu)\,x^2-360\,x^4\right)}{24\,x^2\,(x+1)^2\,(x-1)^2}\,y' - \frac{16}{x\,(x+1)^4\,(x-1)^4}\,y
\end{aligned}
\tag{23}
$$

This equation has two regular singularities at $\{0,\infty\}$ and two irregular singularities at $\{-1,1\}$. Following the steps mentioned in the Summary, we rewrite the equation in normal form and, in step 2.(a), compute the value of k leading to an equation with minimal degrees entering $J_n(x)$ in (15). The value of k found is $k=2$ so the equation from which (23) is derived changing $x \to x^2$ is

$$
\begin{aligned}
y''' = {} & \frac{(6\,\nu+2\,\mu+73-144\,x)}{24\,x\,(x-1)}\,y'' \\
& - \frac{\left(2\,(\nu+8)\,(\mu+1/2)-(48\,\nu+16\,\mu+584)\,x+576\,x^2\right)}{96\,x^2\,(x-1)^2}\,y' - \frac{2}{x^2\,(x-1)^4}\,y
\end{aligned}
\tag{24}
$$

and has invariants with minimal degrees with respect to power transformations. In step 2.(b), analyzing the structure of singularities of (24) we find one regular singularity at the origin and one irregular at ∞. Using the classification discussed in section 3.2 based on the degrees with respect to x of the numerators and denominators of the invariants of (24) as well as the factors entering these denominators the equation is identified as equivalent to the ${}_0\mathrm{F}_2$ class under Möbius transformations (6). So we proceed with step 2.(c), constructing the Möbius transformation and computing the values of the hypergeometric parameters $\{\mu,\nu\}$ entering the ${}_0\mathrm{F}_2$ equation in (3) such that the equivalence under Möbius exists, obtaining:

$$
\alpha = \nu/4+2, \quad \beta = \mu/12+1/24, \quad M := x \to \frac{2\,x}{x-1}
\tag{25}
$$

Composing M above with the power transformation used to obtain (24) and using the values above for α and β, in step 4 we obtain the solution of (23)

$$
\begin{aligned}
y(x) \;=\; & {}_0\mathrm{F}_2(\;;\nu/4+2,\mu/12+1/24;\,2\,\frac{x^2}{x^2-1})\,C_1 \\
& + x^{-(2+\nu/2)}\left(x^2-1\right)^{(1+\nu/4)}\,{}_0\mathrm{F}_2(\;;-\nu/4,\mu/12-\nu/4-23/24;\,\frac{2\,x^2}{x^2-1})\,C_2 \\
& + x^{(23/12-\mu/6)}\left(x^2-1\right)^{(\mu/12-23/24)}\,{}_0\mathrm{F}_2(\;;47/24-\mu/12,\,71/24-\mu/12+\nu/4;\,\frac{2\,x^2}{x^2-1})\,C_3
\end{aligned}
\tag{26}
$$

Meijerg functions and equivalence under rational transformations for the ${}_1\mathrm{F}_2$ class

Consider the following equation, with no symbolic parameters and only integer powers

$$
\begin{aligned}
y''' \;=\; & -\frac{\left(6+12\,x-15\,x^2-6\,x^3\right)}{x\,(1+x-x^2)\,(x+2)}\,y'' \\
& + \frac{\left(16+48\,x+36\,x^2-20\,x^3+9\,x^4+81\,x^5-20\,x^6-30\,x^7-6\,x^8\right)}{x^4\,(x+2)^2\,(1+x-x^2)^2}\,y' \\
& - \frac{(x+2)^3}{(1+x-x^2)^2\,x^5}\,y
\end{aligned}
\tag{27}
$$

Following steps 1 and 2 in the Summary, we confirm that there exists no equivalence under (5), so in step 3 we search for a rational transformation minimizing the degrees of the invariants (16), finding

$$R(x) = x^2/(1+x) \tag{28}$$

Therefore (27) can be obtained by changing variables $x \to R(x)$ in

$$y''' = -\frac{\left(3 - 9\,x + 6\,x^2\right)}{x\,(x-1)^2}\,y'' + \frac{\left(1 - 2\,x + 6\,x^2 - 6\,x^3\right)}{x^3\,(x-1)^2}\,y' - \frac{1}{(x-1)^2\,x^4}\,y \tag{29}$$

This equation[11] thus has invariants with minimal degrees, and has one regular singularity at 1 and one irregular at the origin. According to the classification in terms of singularities (29) admits an equivalence under Möbius transformations to the ${}_p\mathrm{F}_q$ equations (${}_1\mathrm{F}_2$ case) and hence is solved in the iteration step 3.(a) mentioned in the summary. When constructing the ${}_p\mathrm{F}_q$ solutions to (29), however, we find that the ${}_1\mathrm{F}_2$ parameters in the second list are both equal to 1, so only one ${}_1\mathrm{F}_2$ solution is available, and hence two of the MeijerG alternative solutions presented in the table (5) are necessary, resulting in

$$y = {}_0\mathrm{F}_1(\ ;\ 1;\ \frac{1+x-x^2}{x^2})\,C_1 + G^{2,0}_{0,2}\left(\frac{1+x-x^2}{x^2}, \Big|\,{}_{0,0}\right)\,C_2 + G^{3,1}_{1,3}\left(\frac{x^2-x-1}{x^2}, \Big|\,{}^{0}_{0,0,0}\right)\,C_3 \tag{30}$$

Note that the first ${}_p\mathrm{F}_q$ function is a ${}_0\mathrm{F}_1$. This is due to the automatic simplification of order that happens when identical parameters are present in both lists of a ${}_1\mathrm{F}_2$ function; this ${}_0\mathrm{F}_1$ can also be expressed in terms of Bessel functions.

Conclusions

In this work we presented a decision procedure for third order linear ODEs for computing three independent solutions even when they are not Liouvillian or when the hypergeometric parameters involved are such that only two or one ${}_p\mathrm{F}_q$ solution around the origin exists. This algorithm solves complete ODE families we didn't know how to solve before.

The strategy used is that of resolving an equivalence problem to the ${}_3\mathrm{F}_2$, ${}_2\mathrm{F}_2$, ${}_1\mathrm{F}_2$ and ${}_0\mathrm{F}_2$ equations, and in doing so, two important generalizations of the algorithm presented in [9] were developed. First, the classification according to singularities and the use of power composed with Möbius transformations, presented in [9] for 2nd order equations, was generalized for third order ones. Second, the idea of resolving the equivalence mapping into an equation with invariants with "minimal degrees under power transformations" was generalized by determining a transformation mapping into an equation having invariants with "minimal degrees under general rational transformations". This permits resolving a much larger class of ${}_p\mathrm{F}_q$ equations, defined by changing variables in (3) using $\{x \to R(x),\ y \to P(x)\,y\}$ where $R(x)$ is a rational function. Symbolic computation routines implementing this algorithm were integrated into the Maple system in 2007.

Since at the core of the algorithm being presented there is the concept of singularities, two natural extensions of this work consist of applying the same ideas to compute solutions for linear ODEs of arbitrary order, where the equivalence can be solved exactly [14], and for second order equations under rational transformations, perhaps generalizing the work by M.Bronstein [8] with regards to ${}_1\mathrm{F}_1$ solutions to compute also ${}_2\mathrm{F}_1$ solutions. Related work is in progress.

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